How to compute the maximal subsemigroups of a finite semigroup in GAP

Wilf Wilson

18$^{\text{th}}$ March 2015

Joint work with Casey Donoven and James Mitchell

Me

- New PhD student in mathematics.
- As an undergraduate I helped to make: `SmallerDegreePartialPermRepresentation` for Citrus.
    - c.f. `SmallerDegreePermRepresentation` in GAP library.
- My PhD will involve improving computational semigroup theory.

This was a useful function to develop

The `MaximalSubsemigroups` methods apply to all types of semigroup.
The methods use a lot of the functionality in Semigroups package.

Testing `MaximalSubsemigroups` helped highlight issues in the package.

Maximal subgroups and maximal subsemigroups

Definition (maximal subgroup)

Let G be a group and let H be a subgroup of G. Then H is *maximal* if:

- $H \neq G$.
- For all subgroups U:
  $H \leq U \leq G \Rightarrow U = G$ or $U = H$.

Maximal subgroups and maximal subsemigroups

Definition (maximal subgroup)

Let G be a group and let H be a subgroup of G. Then H is *maximal* if:

- $H \neq G$.
- For all subgroups U:
  $H \leq U \leq G \Rightarrow U = G$ or $U = H$.

Definition (maximal subsemigroup)

Let S be a semigroup and let T be a subsemigroup of S. Then T is *maximal* if:

- $T \neq S$.
- For all subsemigroups U:
  $T \leq U \leq S \Rightarrow U = S$ or $U = T$.

A more practical definition (computationally)

Definition (maximal subsemigroup)

Let S be a semigroup and let $T \leq S$. Then T is *maximal* if:

- $S \neq T$.
- For all $x \in S \setminus T$: $\langle T, x \rangle = S$.

A more practical definition (computationally)

Definition (maximal subsemigroup)

Let S be a semigroup and let $T \leq S$. Then T is *maximal* if:

- $S \neq T$.
- For all $x \in S \setminus T$: $\langle T, x \rangle = S$.

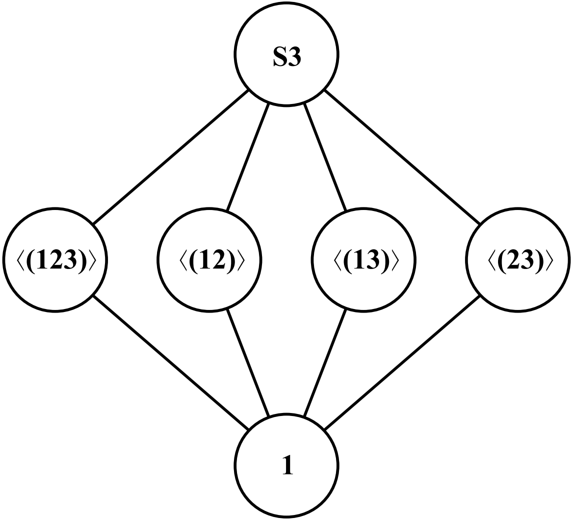
We use this definition in the function `IsMaximalSubsemigroup(S, T)`.

```
return S <> T
       and ForAll(S, x -> x in T or Semigroup(T, x) = S);
```

More sophisticated algorithms did not prove faster.
However in HPC-GAP this could become useful again.

The maximal subgroups of S_3

Let $G = S_3 = \langle (12), (123) \rangle$.



The maximal subsemigroups of a finite group

Let G be a finite group. The subsemigroups of G are the subgroups of G.

Therefore the maximal subsemigroups of G are its maximal subgroups.

The maximal subsemigroups of a finite group

Let G be a finite group. The subsemigroups of G are the subgroups of G.

Therefore the maximal subsemigroups of G are its maximal subgroups.

Therefore we need to calculate maximal subgroups! (Of course!)
This is done very well with GAP: `MaximalSubgroups`.

Green's relations of a semigroup

These are equivalence relations defined on the set S as follows:

Green's relations of a semigroup

These are equivalence relations defined on the set S as follows:

- $x\mathscr{R}y$ if and only if $xS^1 = yS^1$.
- $x\mathscr{L}y$ if and only if $S^1x = S^1y$.
- $x\mathscr{H}y$ if and only if $x\mathscr{R}y$ and $x\mathscr{L}y$.

- $x\mathscr{J}y$ if and only if $S^1xS^1 = S^1yS^1$.

Implemented in the GAP library. e.g. `RClasses(S)`.
Expanded upon in the Semigroups package.

The diagram of a semigroup

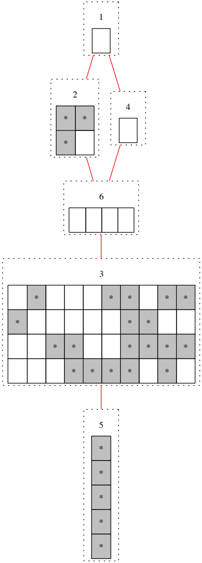
The diagram of the semigroup S generated by these three transformations:

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 5 & 3 \end{pmatrix}$,

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 4 & 1 & 1 \end{pmatrix}$,

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 5 & 2 & 5 & 5 \end{pmatrix}$.

Created by DotDClasses in Semigroups package.



The principal factor J^*

For a $\mathscr{J}$-class J, define J^* to be the semigroup $J \cup \{0\}$, with:

$$x * y = \begin{cases} xy & \text{if } x, y, xy \in J. \\ 0 & \text{otherwise.} \end{cases}$$

Then J^* is isomorphic to a Rees 0-matrix semigroup.

Can calculate J^* easily with Semigroups: `PrincipalFactor(J)`.

The paper that inspired this algorithm

Graham, N. and Graham, R. and Rhodes J.
Maximal Subsemigroups of Finite Semigroups.
Journal of Combinatorial Theory, 4:203-209, 1968.

- Ron Graham wrote Concrete Mathematics with Knuth and Patashnik.

Some useful results from Graham, Graham and Rhodes

Let M be a maximal subsemigroup of a finite semigroup S.

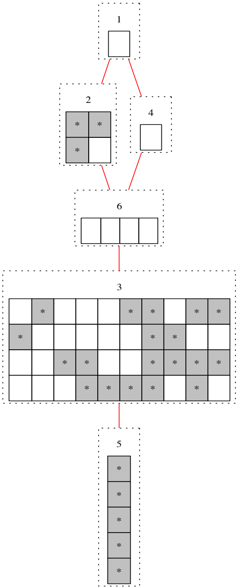
Some useful results from Graham, Graham and Rhodes

Let M be a maximal subsemigroup of a finite semigroup S.

1. M contains all but one $\mathscr{J}$-class of S, J
2. Other conditions...
3. $M \cap J$ corresponds to a special type of subsemigroup of J^*

This is back-to-front!

The diagram of a semigroup (again)



We need to consider J^* for each relevant $\mathscr{J}$-class.
These are independent $\Rightarrow$ parallelisable.

The essential problem is to be able to calculate maximal subsemigroups of Rees 0-matrix semigroups.

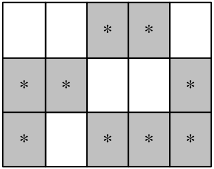
Maximal subsemigroups of a Rees 0-matrix semigroup

Theory tells us to get a maximal subsemigroup we must either:

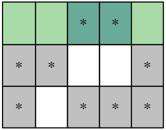
- Replace the group by a maximal subgroup.
- Remove a whole row/column of the semigroup.
- Remove the complement of a maximal rectangle of zeroes.

(With certain conditions).

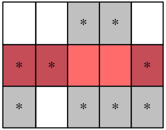
Removing a row...



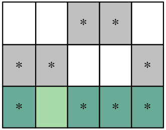
The egg-box diagram of J



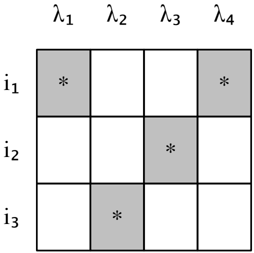
Row 1



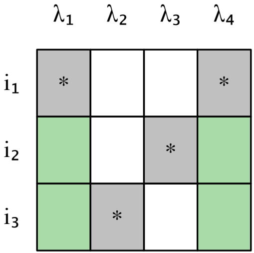
Row 2



Row 3



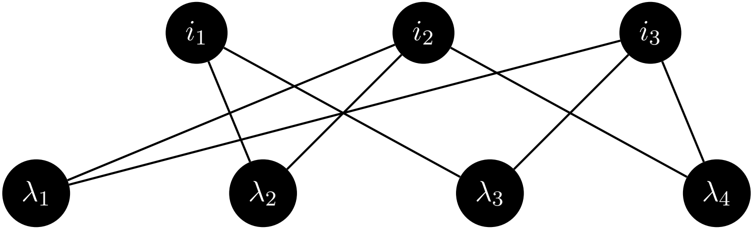
Egg-box diagram



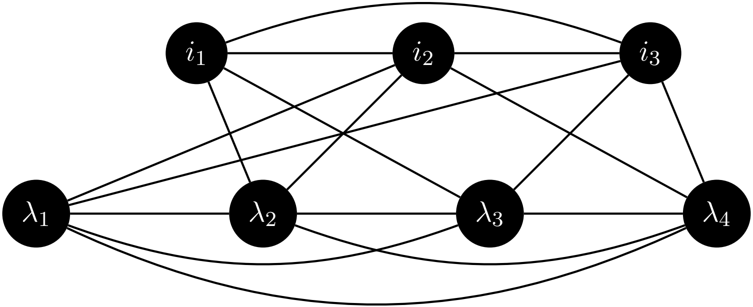
Maximal rectangle

The problem: Find maximal $I' \subset I$ and $\Lambda' \subset \Lambda$ such that $I' \times \Lambda'$ contains only white boxes.

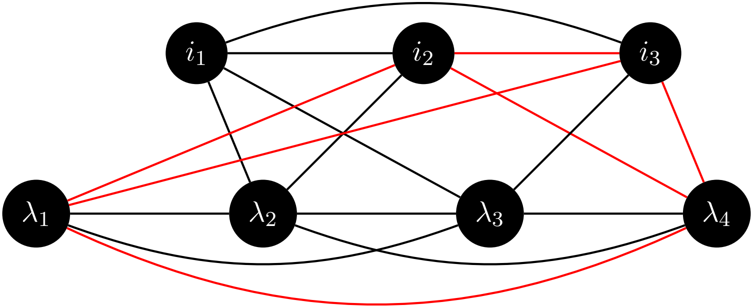
Create a graph...



Add in these extra edges...



Identify maximal cliques...



We use `CompleteSubgraphs` in the GRAPE package.
Could this benefit from HPC-GAP?

End.