Calculating an element of minimal rank in a finite transformation semigroup

Wilf Wilson
University of St Andrews

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Based on idea by Peter Cameron
(via, and with, James Mitchell)

Semigroup

Definition

A semigroup $(S, *)$ is a set S with an associative binary operation $*$ on S. We usually just refer to the semigroup as S.

Combinatorial problems crop up all the time in the study of semigroups!

Computational semigroup theory

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- I implement my ideas in the Semigroups package for the computer algebra system GAP.

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A **transformation** of a finite set Ω is a function $f : \Omega \to \Omega$.

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The composition $f \circ g$ of two transformations f and g is defined as usual:

$$(x)f \circ g = ((x)f)g \text{ for all } x \in \{1, 2, \ldots, n\}.$$

This is associative, so we can consider *transformation semigroups*.

Permutations vs. transformations

We can write permutations in two-line notation:

- e.g. $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$

And we can do the same for transformations:

- e.g. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 3 & 1 \end{pmatrix}$

Permutations vs. transformations

In group theory we have permutation groups, and Cayley's theorem.

- e.g. S_n.

We semigroups we have transformation semigroups and an analogue.

- e.g. T_n.

The kernel and the image of a transformation, f

The **image** of f, $\mathrm{im}(f)$, is the set $\{(i)f : i \in \{1, 2, \ldots, n\}\}$.

The **kernel** of f, $\ker(f)$, is the equivalence relation on $\{1, 2, \ldots, n\}$ which relates i and j whenever $(i)f = (j)f$.

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For a permutation g, $\mathsf{im}(g) = \{1, 2, \ldots, n\}$ and $\mathsf{ker}(g)$ is equality.

The rank of a transformation, f

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Equivalently:

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A transformation on n points has a rank somewhere between 1 and n.

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For a pair $\{i, j\}$ we say that g *collapses the pair* $\{i, j\}$ if $(i)g = (j)g$.

g collapses some pair in $\text{im}(f)$ if and only if $\text{rank}(fg) < \text{rank}(f)$

The kernel and image of a transformation: an example

Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 2 & 1 & 6 & 1 & 7 & 6 & 2 \end{pmatrix}$

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- $\ker(f)$ has equivalence classes:

$$\begin{aligned} \{4, 6\} &= (1)f^{-1}, \\ \{1, 3, 9\} &= (2)f^{-1}, \\ \{2\} &= (5)f^{-1} \\ \{5, 8\} &= (6)f^{-1}, \\ \text{and } \{7\} &= (7)f^{-1}. \end{aligned}$$

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- $\mathrm{rank}(f) = 5$
- f collapses the pairs $\{4, 6\}$, $\{1, 3\}$, $\{1, 9\}$, $\{3, 9\}$, $\{5, 8\}$.

Aim

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For example:

- To see if the semigroup is synchronising (remember Artur's talk?).
- To calculate the zero of a semigroup (or prove it doesn't exist).
- To calculate the elements of the minimal ideal.

Ways of calculating

- **BAD**: getting all of the elements and looking at their ranks in turn.
  - Exponential complexity in n.
- **BETTER**: using the ideas I'm about to share.
  - Quadratic complexity in n.
  - Original ideal from Peter Cameron.
  - Adapted with James Mitchell (my supervisor).

An example

Let $S = \langle \sigma, \tau \rangle$ where:

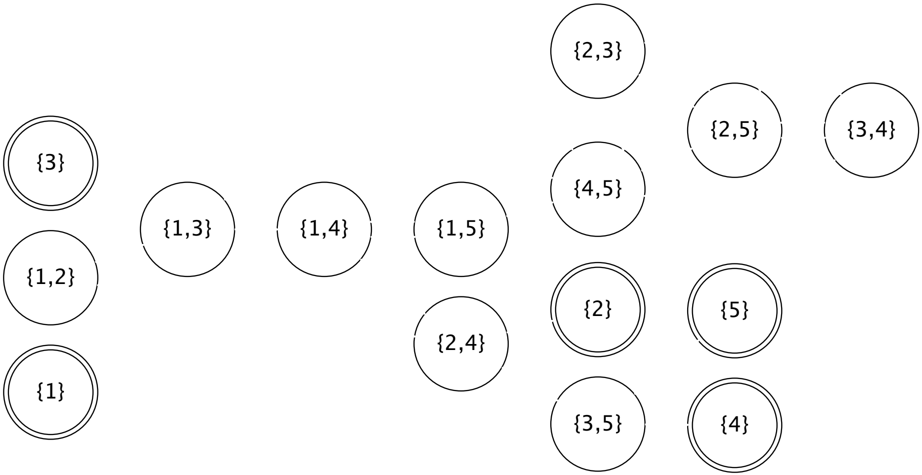
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 4 \end{pmatrix}$$

and

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 2 & 5 \end{pmatrix}$$

are transformations on 5 points.

Graph: the $\binom{5}{2} + 5$ vertices $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 2 & 5 \end{pmatrix}$, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 4 \end{pmatrix}$



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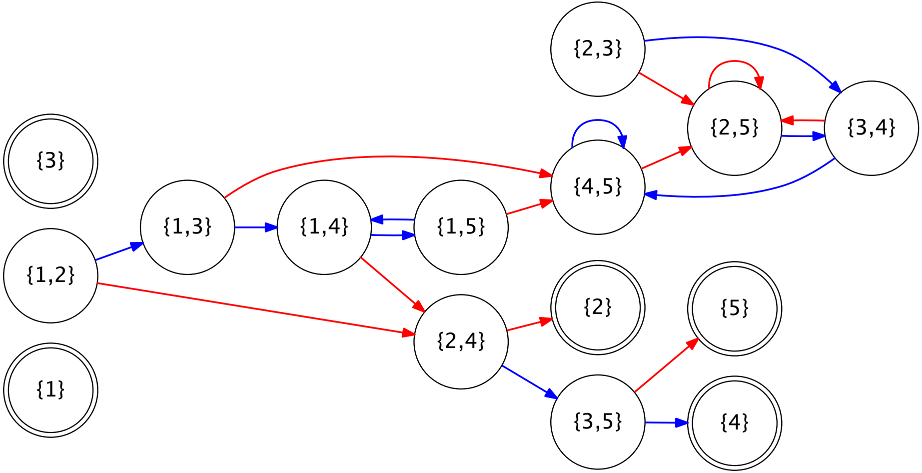
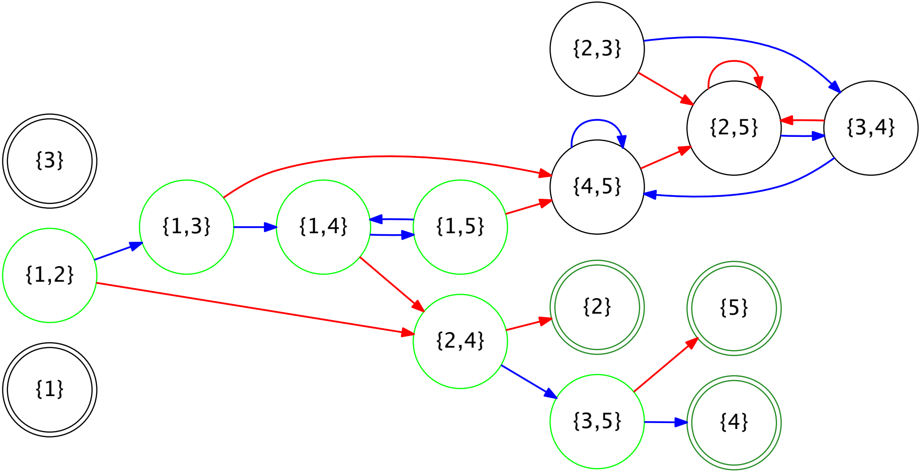


The 6 collapsible pairs $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 2 & 5 \end{pmatrix}$, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 4 \end{pmatrix}$



The two types of pairs $\tau = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 2 & 5 \end{smallmatrix}\right)$, $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 4 \end{smallmatrix}\right)$

The $\binom{5}{2} = 10$ pairs fall into two types: those which can be collapsed, and those which can not.

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Our collapsible pairs:

$$\begin{aligned} \{3,5\}\sigma &= \{4\} \\ \{2,4\}\tau &= \{2\} \\ \{1,2\}\tau^2 &= \{2\} \\ \{1,4\}\tau^2 &= \{2\} \\ \{1,3\}\sigma\tau^2 &= \{2\} \\ \{1,5\}\sigma\tau^2 &= \{2\} \end{aligned}$$

The other pairs:

$$\{2,3\}$$
$$\{2,5\}$$
$$\{3,4\}$$
$$\{4,5\}$$

Every element in S must have different images for i and j if $\{i,j\}$ is not collapsible.

The idea

S must have an element x with minimal rank r.

- Take an element $f \in S$.
- Then $f \circ x$ also has minimal rank.

$\Rightarrow$ We can multiply any non-minimal f by something to decrease its rank.
$\Rightarrow$ For any non-minimal f, there is a collapsible pair of points in im(f).

1. Start with any $f \in S$.
2. Collapse pairs until you can't any more.
3. You now have an element of the minimal ideal.

Let's do it

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No collapsible pairs $\Rightarrow r_2$ minimal.

End.