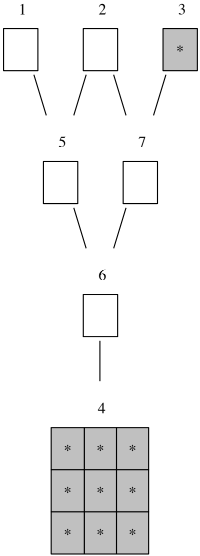
Maximal subsemigroups via independent sets

Wilf Wilson

26th April 2017



- Maximal subsemigroups?
- A maximal subsemigroup is formed by removing parts of *one* $\mathscr{D}$-class.
- It has one of several forms.
- However:
  - A semigroup acts on itself.
  - Elements can generate parts of lower $\mathscr{D}$-classes.
- These things limit the maximal subsemigroups that occur.

My contributions

C. Donoven, J. D. Mitchell, and W. A. Wilson
Computing maximal subsemigroups of a finite semigroup
arXiv:1606.05583

J. East, J. Kumar, J. D. Mitchell, and W. A. Wilson
Maximal subsemigroup of finite transformation and partition monoids
In preparation

The general technique

Focus on some 'nice' monoids!

To find the maximal subsemigroups from a $\mathscr{D}$-class:

- Construct a graph that captures the action on $\mathscr{L}$-/$\mathscr{R}$-classes.
- Compute the maximal independent subsets.
- Find the vertices that are *not* adjacent to a vertex of degree 1.

Partial transformations

Reminders:

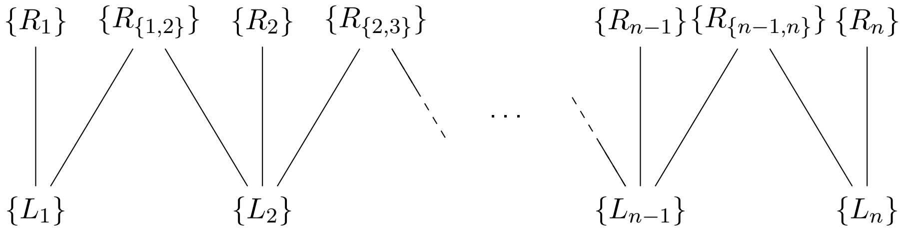
- A *partial transformation of degree* n is a partial map on $\{1, \ldots, n\}$.
- A partial transformation has a *domain*, a *kernel*, and an *image*.
- A total transformation has domain $\{1, \ldots, n\}$.
- Order-preserving: $i \leq j$ implies $(i)f \leq (j)f$.
- Order-reversing: $i \leq j$ implies $(i)f \geq (j)f$.

Notation for Green's classes of rank $n-1$:

- L_i $\mathscr{L}$-class of elements with image $\{1, \ldots, n\} \setminus \{i\}$.
- R_i $\mathscr{R}$-class of elements with domain $\{1, \ldots, n\} \setminus \{i\}$.
- $R_{\{i,j\}}$ $\mathscr{R}$-class of elements with non-trivial kernel class $\{i, j\}$.

Order-preserving partial transformations

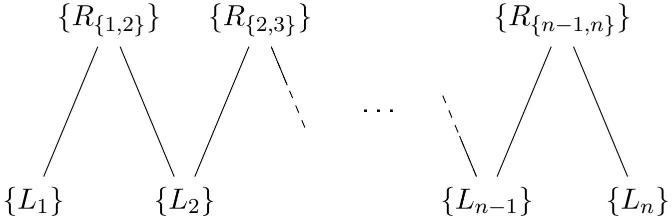
$$|\mathcal{PO}_n| = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k-1}{k}$$



The graph $\Delta(\mathcal{PO}_n)$ has 2^n maximal independent subsets.
$\mathcal{PO}_n$ has $2^n + 2n - 2$ maximal subsemigroups.

Order-preserving transformations

$$|\mathcal{O}_n| = \binom{2n-1}{n}$$



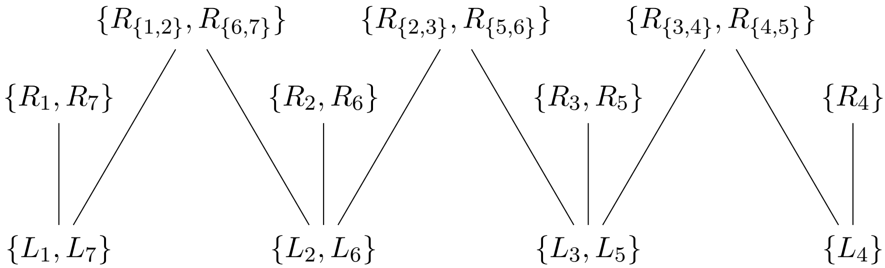
The graph $\Delta(\mathcal{O}_n)$ has A_{2n-1} maximal independent subsets:

$$A_1 = 1,\ A_2 = A_3 = 2,\ \text{and } A_k = A_{k-2} + A_{k-3} \text{ for } k > 3.$$

$\mathcal{O}_n$ has $A_{2n-1} + 2n - 4$ maximal subsemigroups.

Order-preserving or -reversing partial transformations

$$|\mathcal{POD}_n| = 2|\mathcal{PO}_n| - n(2^n - 1) - 1$$



The graph $\Delta(\mathcal{POD}_n)$ has $2^{\lceil n/2 \rceil}$ maximal independent subsets.
$\mathcal{POD}_n$ has $2^{\lceil n/2 \rceil} + n - 1$ maximal subsemigroups.

The Jones monoid

$$|\mathcal{J}_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$$

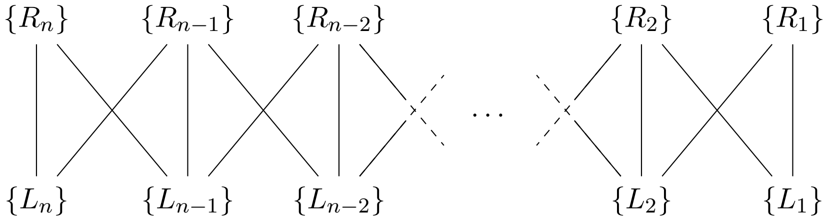


Figure: The graph $\Delta(\mathcal{J}_{n+1})$.

The graph $\Delta(\mathcal{J}_{n+1})$ has $2F_n$ maximal independent subsets.
$\mathcal{J}_{n+1}$ has $2F_n + 2n - 1$ maximal subsemigroups.

Summary: we've replicated previous results, and proved many new ones.