Computing direct products of semigroups

York Semigroup
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Wilf Wilson

University of St Andrews

My motivation for studying direct products

- The SEMIGROUPS package for GAP.

What SEMIGROUPS did really well

```
gap> T4 := FullTransformationMonoid(4);
<full transformation monoid of degree 4>
gap> DirectProduct(T4, T4, T4, T4);
<transformation monoid of size 4294967296,
 degree 16 with 12 generators>
gap> time;
2
gap> T5 := FullTransformationMonoid(5);
<full transformation monoid of degree 5>
gap> DirectProduct(T5, T5, T5, T5, T5);
<transformation monoid of size 298023223876953125,
 degree 25 with 15 generators>
gap> time;
1
```

What SEMIGROUPS didn't do so well

```
gap> Sing3 := SingularTransformationMonoid(3);
<regular transformation semigroup ideal of
 degree 3 with 1 generator>
gap> DirectProduct(Sing3, Sing3, Sing3);
<transformation semigroup of size 9261, degree 9
 with 53 generators>
gap> time;
2245
gap> Sing4 := SingularTransformationMonoid(4);
<regular transformation semigroup ideal of
 degree 4 with 1 generator>
gap> DirectProduct(Sing4, Sing4);
<transformation semigroup of size 53824, degree 8
 with 53 generators>
gap> time;
4507
```

What SEMIGROUPS couldn't do

```
gap> Sing5 := SingularTransformationMonoid(5);
<regular transformation semigroup ideal of
 degree 5 with 1 generator>
gap> DirectProduct(Sing5, Sing5);
gap> P := PartitionMonoid(5);
<regular bipartition *-monoid of size 115975,
 degree 5 with 4 generators>
gap> DirectProduct(P, P);
Error, no method found! For debugging hints type ?Reco
very\
 from NoMethodFound
Error, no 1st choice method found for `DirectProduc\
tOp' on 2 arguments at /Users/Wilf/GAP/lib/methsel2.g:
250 called from
DirectProductOp( arg, arg[1]
 ) at /Users/Wilf/GAP/lib/gprd.gi:27 called from
<function "DirectProduct">( <arguments> )
 called from read-eval loop at *stdin*:68
you can 'quit;' to quit to outer loop, or
you can 'return;' to continue
brk>
```

What do we want?

We want to be able to create the direct product of *‘any’* collection of finite semigroups.

We want this to perform in a way that:

- terminates *‘reasonably’* quickly;
- gives a *‘reasonably’* small generating set; and
- uses a *‘reasonable’* representation.

What's the situation for groups?

Let $G = \langle X \rangle$ and $H = \langle Y \rangle$ be groups.

Then $G \times H$ is generated by:

$$\{(1_G, h) : h \in Y\} \cup \{(g, 1_H) : g \in X\}.$$

Therefore,

$$\operatorname{rank}(G \text{ or } H) \leqslant \operatorname{rank}(G \times H) \leqslant \operatorname{rank}(G) + \operatorname{rank}(H).$$

Examples:

$$\operatorname{rank}(\mathcal{C}_2 \times \mathcal{C}_2) = 2 \quad \text{and} \quad \operatorname{rank}(\mathcal{C}_2 \times \mathcal{C}_3) = \operatorname{rank}(\mathcal{C}_6) = 1.$$

(Same idea for monoids.)

What's the situation for groups?

Define $\mu(G) = \min\left\{n \in \mathbb{N} : G \hookrightarrow \mathcal{S}_n\right\}$, the minimal degree of a permutation representation of G. Then

$$\mu(G \times H) \leqslant \mu(G) + \mu(H),$$

and often equality holds.

$\mu(\mathcal{C}_p) = p$: $\langle(1\ 2\ \ldots\ p)\rangle$.

$\mu(\mathcal{C}_2 \times \mathcal{C}_2) = 4$: $\langle(1\ 2)\rangle \times \langle(1\ 2)\rangle \cong \langle(1\ 2),\ (3\ 4)\rangle$.

$\mu(\mathcal{C}_2 \times \mathcal{C}_3) = 5$: $\langle(1\ 2)\rangle \times \langle(1\ 2\ 3)\rangle \cong \langle(1\ 2),\ (3\ 4\ 5)\rangle$.

So far, so easy?

We can't pretend that every semigroup is a monoid.
Yes, $S \times T$ embeds in $S^1 \times T^1$, but that doesn't help us!

If $(s, t) \in S \times T$ and $S \times T = \langle X \rangle$, then

$$(s, t) = (s_1, t_1)(s_2, t_2) \cdots (s_m, t_m)$$

for some generators $(s_i, t_i) \in X$.

- $s = s_1 \cdots s_m$ and
- $t = t_1 \cdots t_m$ and
- (s_i, t_i) is a generator for each i.

What can go wrong

The natural numbers with addition are monogenic...
But $\mathbb{N} \times \mathbb{N}$ is not finitely generated!

This is because 1 is not the sum of two naturals.

If $(1, n) = (s_1, t_1) \cdots (s_m, t_m) \in \mathbb{N} \times \mathbb{N}$, then

$$1 = s_1 + \cdots + s_m.$$

Therefore $m = s_1 = 1$, and $(1, n)$ is a generator.

Decomposable and indecomposable elements

Let S be a semigroup, and let $s \in S$.

- s is *decomposable* if $s \in S^2$.
- s is *indecomposable* if $s \in S \setminus S^2$.
- S is *decomposable* if $S = S^2$.

Straightforward results:

- Any generating set for S contains $S \setminus S^2$.
- $(s_1, s_2, \ldots, s_n)$ is decomposable $\Leftrightarrow$ each s_i is.
- $(s_1, s_2, \ldots, s_n)$ is indecomposable $\Leftrightarrow$ any s_i is.
- $S_1 \times S_2 \times \cdots \times S_n$ is decomposable $\Leftrightarrow$ each S_i is.
- Any generating set for $S_1 \times \cdots \times S_n$ contains

$$S_1 \times \cdots \times S_{i-1} \times (S_i \setminus S_i^2) \times S_{i+1} \times \cdots \times S_n.$$

Decomposable semigroups

Suppose that $S = S^2 = \langle X \rangle$, and let $x \in X$.

$$\begin{aligned} \text{Then } x &= x_1 \cdots x_{n-1} x_n && \text{for some } x_i \in X, n \geqslant 2. \\ &= a_x x_n && \text{where } a_x = x_1 \cdots x_{n-1}. \end{aligned}$$

Define $A_X = \{a_x : x \in X\}$. Therefore $X \subseteq A_X X$.

We can *extend* the length of any product in X:

- If $s = x_1 x_2 x_3$, then $s = x_1 (a_{x_2} x') x_3$ for some $x' \in X$.

Similarly, if $T = T^2 = \langle Y \rangle$, define A_Y. Then $Y \subseteq A_Y Y$.

If $(s, t) \in S \times T$, then for some $k \in \mathbb{N}$,

$$s \in (A_X \cup X)^k \qquad \text{and} \qquad t \in (A_Y \cup Y)^k.$$

Generators for decomposable direct products

Theorem (Robertson, Ruškuc, Wiegold, 1998)

Let S and T be decomposable semigroups. Then $S \times T$ is generated by

$$(A_X \times Y) \cup (A_X \times A_Y) \cup (X \times A_Y) \cup (X \times Y).$$

Corollary (ibid.)

Let S and T be decomposable semigroups. Then

$$\operatorname{rank}(S \times T) \leqslant 4 \operatorname{rank}(S) \operatorname{rank}(T).$$

And an open problem: *is this best possible?*

An improvement?

- This construction seems to be a bit wasteful:
  - We only ever expand either s or t in (s, t).
  - We arbitrarily chose to expand with $x \mapsto ax'$.
- So can we do better? Yes.

Decomposable semigroups, again

Suppose that $S = S^2 = \langle X \rangle$, and let $x \in X$.

$$\begin{aligned} \text{Then } x &= x_1 \cdots x_{n-1} x_n && \text{for some } x_i \in X, n \geqslant 2. \\ &= a_x x_n && \text{where } a_x = x_1 \cdots x_{n-1}. \end{aligned}$$

Define $A_X = \{ a_x : x \in X \}$. Therefore $X \subseteq A_X X$.

Suppose that $T = T^2 = \langle Y \rangle$, and let $y \in Y$.

$$\begin{aligned} \text{Then } y &= y_1 y_2 \cdots y_n && \text{for some } y_i \in Y, n \geqslant 2. \\ &= y_1 b_y && \text{where } b_y = y_2 \cdots y_n. \end{aligned}$$

Define $B_Y = \{ b_y : y \in Y \}$. Therefore $Y \subseteq B_Y Y$.

Then $(s, t) \in (A_X \times Y)^m (X \times B_Y)^n$ for some $m, n \in \mathbb{N}$.

An improvement!

Theorem (Isabel Araújo, PhD thesis, 2000)

Let S and T be decomposable semigroups. Then $S \times T$ is generated by

$$(A_X \times Y) \cup (X \times B_Y).$$

Corollary (ibid.)

Let S and T be decomposable semigroups. Then

$$\operatorname{rank}(S \times T) \leqslant 2 \operatorname{rank}(S) \operatorname{rank}(T).$$

(And this is best possible, in general.)

Induction. . .

Let $S_1, \ldots, S_n$ be decomposable semigroups. Then

$$\operatorname{rank}(S_1 \times \cdots \times S_n) \leqslant 2^{n-1} \cdot \prod_{i=1}^{n} \operatorname{rank}(S_i).$$

Finite generation of direct products

Theorem (Isabel Araújo, PhD thesis, 2000)

Let $S_1, S_2, \ldots, S_n$ be a collection of semigroups. Then

$$S_1 \times S_2 \times \cdots \times S_n$$

is finitely generated if and only if

- *each S_i is finitely generated;*

and

1. *S_i is finite for all i; or*
2. *S_i is decomposable for all i; or*
3. *S_j infinite, S_i is finite & decomposable for all $i \neq j$.*

But what about generating sets for direct products:

- of more than two decomposable semigroups?
- where not all factors are decomposable?

My improvement

For each $i \in \{1, \ldots, n\}$, let $S_i = \langle X_i \rangle$ be a semigroup, and define A_i and B_i so that

$$(S_i^2 \cap X_i) \subseteq (A_i X_i) \cap (X_i B_i).$$

Then $S_1 \times \cdots \times S_n$ is generated by:

$$\bigcup_{i=1}^{n} \Big(\big(B_1 \times \cdots \times B_{i-1} \times (S_i^2 \cap X_i) \times A_{i+1} \times \cdots \times A_n\big) \cup \\ \cup \big(S_1 \times \cdots \times S_{i-1} \times (S_i \setminus S_i^2) \times S_{i+1} \times \cdots \times S_n\big) \Big).$$

Corollaries

Let $S_1, \ldots, S_n$ be decomposable semigroups. Then

$$\operatorname{rank}(S_1 \times \cdots \times S_n) \leqslant n \cdot \prod_{i=1}^{n} \operatorname{rank}(S_i).$$

Let $S_1, \ldots, S_n$ be semigroups and suppose $S_j = \langle S_j \setminus S_j^2 \rangle$ for some j. Then

$$\bigcup_{i=1}^{n} \big(S_1 \times \cdots \times S_{i-1} \times (S_i \setminus S_i^2) \times S_{i+1} \times \cdots \times S_n\big)$$

is the unique minimal generating set.

Putting this into practice

To construct a generating set, we must:

- Find the indecomposable generators; and
- Construct the sets A_i and B_i.

These steps can be combined.

- Creating A_i and B_i requires non-trivial factorizations of generators over the generators.
- An element has a non-trivial factorization if and only if it is decomposable.

Finding non-trivial factorizations

- Search for paths in the Cayley graph.
- Look at the multiplication table.
- Use the Green's structure of the semigroup.
- A non-trivial factorization of a generator x of a finitely-presented semigroup $\langle X \mid R \rangle$ is a relation of the form $x = w$, for some $w \in XX^+$.

Finding non-trivial factorizations is fairly quick.

What SEMIGROUPS has improved upon

```
gap> Sing3 := SingularTransformationMonoid(3);
<regular transformation semigroup ideal of
 degree 3 with 1 generator>
gap> DirectProduct(Sing3, Sing3, Sing3);
<transformation semigroup of size 9261, degree 9
 with 169 generators>
gap> time;
26
gap> Sing4 := SingularTransformationMonoid(4);
<regular transformation semigroup ideal of
 degree 4 with 1 generator>
gap> DirectProduct(Sing4, Sing4, Sing4, Sing4);
<transformation semigroup of size 2897022976,
 degree 16 with 8392 generators>
gap> time;
107
```

What SEMIGROUPS can now do

```
gap> Sing5 := SingularTransformationMonoid(5);
<regular transformation semigroup ideal of
 degree 5 with 1 generator>
gap> DirectProduct(Sing5, Sing5);
<transformation semigroup of size 9030025,
 degree 10 with 274 generators>
gap> time;
50
gap> DirectProduct(Sing5,Sing5,Sing5,Sing5,Sing5);
<transformation semigroup
 of size 245031761259378125, degree 25 with 534136
 generators>
gap> time;
32494
```

SEMIGROUPS handles more representations

```
gap> P := PartitionMonoid(5);
<regular bipartition *-monoid of size 115975,
 degree 5 with 4 generators>
gap> DirectProduct(P, P);
<bipartition monoid of size 13450200625, degree 10
 with 8 generators>
gap> T5 := FullTransformationMonoid(5);
<full transformation monoid of degree 5>
gap> DirectProduct(T5, P);
<transformation monoid of size 362421875,
 degree 115980 with 7 generators>
```

Minimal transformation representations

Define $\mu(S) = \min\left\{n \in \mathbb{N} : S \hookrightarrow \mathcal{T}_n\right\}$, the minimal degree of a transformation representation of S.

In general, $\mu(S \times T) \leqslant \mu(S) + \mu(T)$.

Also define:

- L_m = left zero semigroup of order m;
- R_n = right zero semigroup of order n;
- $B_{m,n}$ = $m \times n$ rectangular band.

Then:

- $\mu(L_6) = \mu(L_2 \times L_3) = 5 \lneq 7 = 3 + 4 = \mu(L_2) + \mu(L_3)$.
- $\mu(R_{25}) = \mu(R_5 \times R_5) = 9 \lneq 10 = 2 \cdot \mu(R_5)$.
- $\mu(B_{2,2}) = \mu(L_2 \times R_2) = 4 \lneq 5 = 3 + 2 = \mu(L_2) + \mu(R_2)$.

(D. Easdown, D. Easdown, J. D. Mitchell.)

A different direction...

Let $S^2 = S$ and $T^2 = T$ be finitely generated semigroups.

- Define l to be the number of maximal $\mathscr{L}$-classes of S that are regular, and l' to be the number that are not.
- Define r to be the number of maximal $\mathscr{R}$-classes of T that are regular, and r' to be the number that are not.

Then:

$$\operatorname{rank}(S \times T) \leqslant \operatorname{rank}(S) \cdot (r + 2r') + (l + 2l') \cdot \operatorname{rank}(T).$$

(This can be smaller than $2 \operatorname{rank}(S) \operatorname{rank}(T)$*.)*